HEAT CONDUCTION IN AN ELLIPTICAL CYLINDER

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Direct analytical methods are applied to solve the third boundary problem of heat conduction in an elliptical cylinder.

In solving the third boundary problem of heat conduction in hexahedra, the classical method of Lame of separation of all the variables (spatial and temporal) encounters fundamental difficulties even when it is possible to choose a separable system whose coordinate surfaces coincide with a boundary of the body (as, for example, in the case considered here, of the elliptical cylinder). The artificial use of the so-called normal form of the solution involving a series of terms consisting of products of functions, each depending on only one variable, predetermines a specific distribution law for the heat transfer coefficient, and, in particular, a solution with completely separated variables corresponds only to a particular case wherein on each boundary Biot's criterion is inversely proportional to the corresponding Lame coefficient [1]. It is clear that such a dependence of Biot's criterion on the geometrical parameters of the surface is not a consequence of the physical nature of convective heat transfer, and therefore the classical solution of the third boundary problem is of practical validity in a limited number of cases only.

In contrast to this, direct analytical methods are not beset with the fundamental limitations on either the shape of the body or the boundary conditions. Such an extension of the class of solvable problems is attained at the expense of not satisfying precisely the differential equation and boundary conditions. Direct methods yield convergence in the mean over the region (or an exact satisfaction of the equation and boundary conditions at individual points on certain curves and surfaces).

Their application to nonstationary heat conduction is based on Kantorovich's method [2], according to which an approximate solution of the problem

$$l(t) = \frac{\partial t}{\partial \mathrm{Fo}} - R^2\nabla^2 t - \frac{R^2}{\lambda}\, w = 0, \quad (x_1,\ x_2,\ x_3) \in v, \tag{1}$$

$$m(t) = R\,\frac{\partial t}{\partial n} + \mathrm{Bi}\, t - \frac{R}{\lambda}\, p = 0, \quad (x_1,\ x_2,\ x_3) \in \sigma, \tag{2}$$

$$t(0,\ x_1,\ x_2,\ x_3) - t_0 = 0, \quad (x_1,\ x_2,\ x_3) \in v + \sigma \tag{3}$$

is sought in the form of a sum

$$\tilde{t} = \sum_{j=1}^{n} f_j(\mathrm{Fo})\, \varphi_j(x_1,\ x_2,\ x_3) \tag{4}$$

or

$$\tilde{t} = \sum_{j=1}^{n} f_j(\mathrm{Fo})\, \varphi_j(\mathrm{Fo},\ x_1,\ x_2,\ x_3), \tag{5}$$

where the f_j are the desired time functions and the φ_j are a priori selected coordinate functions.

Substitution of $\tilde{t}$ into the differential equation and the boundary condition yields specific residuals. Following Kantorovich we minimize these residuals so that at each instant of time they are orthogonal to each of the coordinate functions throughout the region $v + \sigma$ in which the temperature is defined. This requirement leads to a system of ordinary differential equations. Its solution is the set of desired functions

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$f_j(\mathrm{FO})$. The constants of integration may be determined from the initial condition, minimizing for Fo = 0 the mean square deviation of the solution from the given t_0 distribution (this operation reduces to the solution of a system of algebraic equations).

Such is the general method of solving both linear and nonlinear problems [3]. However in the first case it is possible to shorten substantially the amount of computation by first taking the Laplace transform of Eqs. (1) and (2) with respect to the variable Fo and thus to obtain the boundary problem

$$L(T) = sT - R^2\nabla^2 T - \frac{R^2}{\lambda} W - t_0 = 0, \quad T \doteqdot t, \ W \doteqdot w, \tag{1a}$$

$$M(T) = R\frac{\partial T}{\partial n} + \mathrm{Bi}\, T - \frac{R}{\lambda} P = 0, \quad s \doteqdot \mathrm{Fo}, \ P \doteqdot p. \tag{2a}$$

If from this we determine T as a function of the spatial coordinates and then take the inverse Laplace transform, we obtain the desired nonstationary temperature field.

We seek an approximate solution of Eqs. (1a) and (2a) in the form of a sum

$$\widetilde{T} = \Phi(s, x_1, x_2, x_3) + \sum_{j=1}^{n} F_j(s)\, \varphi_j(x_1, x_2, x_3), \tag{4a}$$

where Φ is an approximate (or exact) solution of Eq. (1a), and the φ_j are coordinate functions satisfying the orthonormality condition

$$\overline{(\varphi_i, \varphi_j)}^v = \delta_{ij}. \tag{5a}$$

To determine the F_j we substitute from Eq. (4a) into Eqs. (1a) and (2a) and apply Galerkin's method, which requires the orthogonality of the residuals $L(\widetilde{T})$ and $M(\widetilde{T})$, so formed, to the coordinate functions φ_i

$$\overline{(L(\widetilde{T}), \varphi_i)}^v + \overline{(M(\widetilde{T}), \varphi_i)}^\sigma = 0. \tag{6}$$

We substitute Eq. (4a) into Eq. (6) and change the orders of summation and integration:

$$\sum_{j=1}^{n} F_j(s) \left\{ \overline{\left((s\varphi_j - R^2\nabla^2\varphi_j), \varphi_i\right)}^v + \overline{\left(\left(R\frac{\partial \varphi_j}{\partial n} + \mathrm{Bi}\,\varphi_j\right), \varphi_i\right)}^\sigma \right\} = a_i + B_i + C_i, \tag{7}$$

where

$$a_i = \overline{(t_0, \varphi_i)}^v; \ B_i = \overline{\left(\frac{R^2}{\lambda} W + R^2\nabla^2\Phi - s\Phi, \varphi_i\right)}^v; \tag{8}$$

$$C_i = \overline{\left(\frac{R}{\lambda} P - n\frac{\partial \Phi}{\partial n} - \mathrm{Bi}\,\Phi, \varphi_i\right)}^\sigma.$$

Using Green's formula

$$\overline{(\nabla^2\varphi_j, \varphi_i)}^v = \frac{1}{R}\overline{\left(\frac{\partial \varphi_j}{\partial n}, \varphi_i\right)}^\sigma - \overline{(\mathrm{grad}\,\varphi_j, \mathrm{grad}\,\varphi_i)}^v,$$

we write Eqs. (7) of Galerkin's method in the symmetric form:

$$\sum_{j=1}^{n} (\delta_{ij}s + \alpha_{ij}) F_j = a_i + B_i + C_i, \tag{9}$$

where

$$\alpha_{ij} = \alpha_{ji} = R^2\,\overline{(\mathrm{grad}\,\varphi_i, \mathrm{grad}\,\varphi_j)}^v + \overline{(\mathrm{Bi}\,\varphi_i, \varphi_j)}^\sigma. \tag{10}$$

Let Δ denote the determinant of the matrix of the coefficients and let Δ_{ij} be the cofactor of the element (i, j). Then the solution of the algebraic system of Eqs. (9) assumes the following form:

$$F_j = \Delta^{-1}\sum_{i=1}^{n} (a_i + B_i + C_i)\,\Delta_{ij}. \tag{11}$$

From this we obtain

$$\widetilde{T} = \Delta^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} (a_i + B_i + C_i)\,\Delta_{ij}\varphi_j. \tag{12}$$

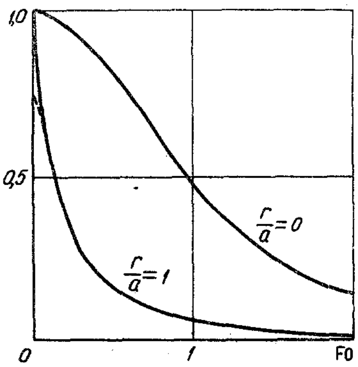

Fig. 1. Variation of the temperature in a cross section of a circular cylinder with Bi = 5. (The continuous curve represents the exact solution; dashed curve, the approximate solution.)

In the inversion of the Laplace transform we use the factorization theorem (which is applicable since the determinant Δ, expanded as a polynomial in s, is of much higher degree than the algebraic cofactor Δ_{ij}) and the convolution theorem. As a result of the transformation we obtain

$$\tilde{t} = \sum_{k=1}^{n} K_k u_k \exp(s_k \mathrm{Fo}) + \sum_{k=1}^{n} K_k \int_0^{\mathrm{Fo}} v_k \exp\left[s_k(\mathrm{Fo} - \vartheta)\right] d\vartheta, \tag{13}$$

where

$$u_k = \sum_{j=1}^{n} \varphi_j \sum_{i=1}^{n} a_i \Delta_{ij}; \quad v_k = \sum_{j=1}^{n} \varphi_j \sum_{i=1}^{n} (b_i + c_i) \Delta_{ij};$$

the s_k are the roots of the secular equation $\Delta(s) = 0$; $K_k^{-1} = d/ds\, \Delta(s_k)$; $b_i \doteq B_i$; $c_i \doteq C_i$. It is obvious that $\sum_{i=1}^{n} a_i \Delta_{ij}$ is the determinant formed from $\Delta(s_k)$ by replacing the elements of its j-th column by the a_i. The second summation in the expression for v_k has a similar meaning, it being in general a function of the time. In the particular case where the specific strength of the volume and surface sources, w and p, depends only on the spatial coordinates,

$$\tilde{t} = \sum_{k=1}^{n} K_k \{\exp(s_k \mathrm{Fo})\,[u_k + s_k^{-1} v_k] - s_k^{-1} v_k\}. \tag{13a}$$

A comparison of the structure of formula (13) with the classical solution shows that the K_k are analogous to the Fourier–Lame coefficients, the u_k are characteristic functions and the s_k are characteristic values.

If as the coordinate functions we choose algebraic or trigonometric polynomials, the convergence of Galerkin's method follows from a theorem due to Weierstrass [2].

In order to obtain a better approximation with a small number of terms of the series (13) it is necessary to select the coordinate functions corresponding to the geometric nature of the problem. To construct these functions it is convenient to apply the process of successive orthogonalization to a system of fundamental functions ψ, satisfying homogeneous boundary conditions of the form (2), if not exactly, then at least approximately:

$$\varphi_1 = N_1^{-1} \psi_1, \quad N_1^2 = \overline{(\psi_1, \psi_1)^v},$$

$$\varphi_j = N_j^{-1} \left[\psi_j - \sum_{k=1}^{j-1} \varphi_k \overline{(\varphi_k, \psi_j)^v}\right], \tag{14}$$

$$N_j^2 = \overline{(\psi_j, \psi_j)^v} - \sum_{k=1}^{j-1} \left[\overline{(\varphi_k, \psi_j)^v}\right]^2.$$

By forming the scalar product $\overline{(\varphi_i, \varphi_j)^V}$, it is easy to verify that the coordinate functions constructed in this way actually satisfy the orthonormality condition (5a).

We now apply the formulas obtained to the calculation of a plane-parallel nonstationary thermal field in an elliptical cylinder with semiaxes a and b. Exact solutions of the first and second boundary problems for an elliptical cylinder may be expressed in terms of Mathieu functions. However, in the case of the third

TABLE 1. First Five Characteristic Values s_k of the Circular Cylinder (Bi = 5)

$-s_k$	k				
	1	2	3	4	5
Approximate	1,1891	6,3515	15,8962	29,5523	51,8124
Exact	1,1875	6,3532	15,8281	29,900	48,7064

TABLE 2. First Five Characteristic Values s_k and Coefficients K_k of the Solutions (13) for the Elliptical Cylinder (Bi = 5)

k	1	2	3	4	5
$-s_k$	1,20599	6,35129	15,71763	31,05346	79,52921
K_k	$-0{,}57284\cdot 10^{-5}$	$-0{,}11479\cdot 10^{-4}$	$+0{,}751808\cdot 10^{-5}$	$-0{,}182442\cdot 10^{-5}$	$+0{,}564034\cdot 10^{-7}$

boundary problem an exact solution (with separation of variables), as has already been noted, can be obtained only for a special form of the dependence of the heat transfer coefficient on the coordinates of the surface of the body. Such a result is of no practical value. It is therefore necessary to use approximate methods of solution. We consider the special case $w = p = 0$; α = const; t_0 = const.

We carry through concrete calculations for two elliptical cylinders of eccentricities $\varepsilon = 0$ and $\varepsilon = 0.8$ (the first example furnishes a comparison with the known exact solutions for a circular cylinder). We put Bi = 5 and n = 5, i.e., we construct five coordinate functions (to save space, however, we write out only the first three of them).

As fundamental functions we select the polynomials

$$\psi_m = \left[\left(P_{m-1}\frac{x}{a}\right)^2 + \left(Q_{m-1}\frac{y}{b}\right)^2\right]^{m-1} - \left[\left(P_m\frac{x}{a}\right)^2 + \left(Q_m\frac{y}{b}\right)^2\right]^m \tag{15}$$

and try to satisfy the homogeneous boundary conditions of the third kind having the form (2a). For the circular cylinder (a = b = 2R) we find directly that

$$P_0 = Q_0 = 1;\ P_m^{2m} = Q_m^{2m} = \mathrm{Bi}\,(\mathrm{Bi} + m)^{-1}.$$

Letting $r^2 = x^2 + y^2$, we obtain the following expressions for the functions ψ:

$$\psi_1 = 1 - \frac{5}{6}\left(\frac{r}{a}\right)^2;\quad \psi_2 = \frac{5}{6}\left(\frac{r}{a}\right)^2 - \frac{5}{7}\left(\frac{r}{a}\right)^4;$$

$$\psi_3 = \frac{5}{7}\left(\frac{r}{a}\right)^4 - \frac{5}{8}\left(\frac{r}{a}\right)^6.$$

From these functions we construct through the successive orthogonalization (14) a set of orthonormal coordinate functions

$$\varphi_1 = 1.5847\,\psi_1,\quad \varphi_2 = 8.8332\,[\psi_2 - 0.24086\,\psi_1],$$

$$\varphi_3 = 36.8011\,[\psi_3 - 0.1256\,\psi_2 - 0.07291\,\psi_1].$$

Next we compute the s_k, the roots of the secular equation $\Delta(s) = 0$, corresponding to the system of algebraic equations of the form (9), and, finally, to construct the final solution (13a), we define the functions u_k and the coefficients K_k. In Table 1, for comparison, we give the exact characteristic values, equal to $0.25\,\mu_k^2$ (in the notation of [4]) and the approximate values s_k.

As is evident from the table, only the fourth and fifth characteristic values display a noticeable error (1.5% and 6.5%). Therefore, already for Fo > 0.1, the approximate solution is close to the exact solution (see Fig. 1).

In the general case of the elliptical cylinder we can no longer, using algebraic polynomials, satisfy the boundary condition of the third kind over the whole lateral surface. Therefore, at the vertices of the semiaxes a and b, we satisfy the boundary condition exactly, and at the remaining points, approximately. This requirement leads to the following formulas

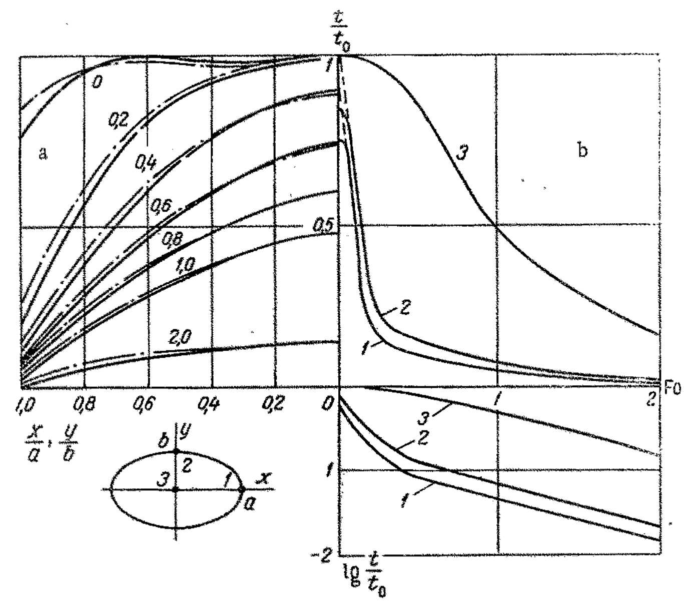

Fig. 2. Variation of the temperature at a section of an elliptical cylinder with Bi = 5: a) variation along the semiaxes (continuous curve for the major axis; dashed curve for the minor axis; Fo values are shown on the curves); b) variation at the vertices of the semiaxes (curves 1, 2, and 3 correspond, respectively, to the vertex of the minor axis, and to the point (0, 0).

$$P_0 = 1,\ P_m^{2m} = \mathrm{Bi}\left(\mathrm{Bi} + m\,\frac{2R}{a}\right)^{-1},$$
$$Q_0 = 1,\ Q_m^{2m} = \mathrm{Bi}\left(\mathrm{Bi} + m\,\frac{2R}{b}\right)^{-1}, \qquad (16)$$

where

$$\frac{a}{R} = \frac{4}{\pi}\,\frac{E(\varepsilon)}{\sqrt{1-\varepsilon^2}},\quad \frac{b}{R} = \frac{4}{\pi}\,E(\varepsilon).$$

The scalar products appearing in formulas (14) and (10) are determined by recursion relations in which the integration over the volume is reduced to the calculation of factorials and that over the cylindrical surface to the calculation of complete elliptic integrals of the first and second kind. To determine the values it is expedient to solve the homogeneous system obtained from the Galerkin equation (9) by an iterational method (the process converges rapidly owing to the predominance of the diagonal elements).

Using Eqs. (15) and (16) we obtain the following expressions for the first three fundamental functions:

$$\psi_1 = 1 - \left[\left(0.93344\,\frac{x}{a}\right)^2 + \left(0.89581\,\frac{y}{b}\right)^2\right],$$
$$\psi_2 = \left[\left(0.93344\,\frac{x}{a}\right)^2 + \left(0.89581\,\frac{y}{b}\right)^2\right] - \left[\left(0.93735\,\frac{x}{a}\right)^2 + \left(0.90477\,\frac{y}{b}\right)^2\right]^2,$$
$$\psi_3 = \left[\left(0.93735\,\frac{x}{a}\right)^2 + \left(0.90477\,\frac{y}{b}\right)^2\right]^2 - \left[\left(0.94070\,\frac{x}{a}\right)^2 + \left(0.91195\,\frac{y}{b}\right)^2\right]^3.$$

Next, in accord with Eqs. (14), we determine the coordinate functions

$$\varphi_1 = 1.58757\,\psi_1,\quad \varphi_2 = 8.84493\,\psi_2 - 2.13055\,\psi_1,$$
$$\varphi_3 = 44.38226\,\psi_3 - 27.94131\,\psi_2 - 10.88060\,\psi_1.$$

Finally, we find the characteristic functions u_k

$$u_1 = 0.15658 \cdot 10^6 \varphi_1 - 0.12003 \cdot 10^5 \varphi_2 + 0.36443 \cdot 10^3 \varphi_3 + 0.55667 \cdot 10^1 \varphi_4 + 0.19076 \cdot 10^2 \varphi_5,$$

$$u_2 = -0.20817 \cdot 10^4 \varphi_1 - 0.30062 \cdot 10^5 \varphi_2 + 0.79359 \cdot 10^4 \varphi_3 - 0.89209 \cdot 10^3 \varphi_4 + 0.36927 \cdot 10^2 \varphi_5,$$

$$u_3 = 0.50617 \cdot 10^2 \varphi_1 + 0.61429 \cdot 10^4 \varphi_2 + 0.22762 \cdot 10^5 \varphi_3 - 0.11495 \cdot 10^5 \varphi_4 + 0.22654 \cdot 10^4 \varphi_5,$$

the characteristic values s_k, and the coefficients K_k (Table 2).

In Fig. 2 we show the variation of the temperature during the cooling of an elliptical cylinder. As was to be expected, the most rapid cooling occurs at the vertex of the major semiaxis. For the value Bi = 5, assumed in the calculations, the regular regime is established beginning with Fo ≈ 0.8.

Since the solutions (13) and (13a) contain a finite number of terms, the initial conditions are satisfied with a certain error (see the curves for Fo = 0 in Fig. 2a). An extrapolation of the thermal curves (Fig. 2b) shows that this error no longer manifests itself after Fo = 0.1.

NOTATION

v	is the volume;
σ	is the surface area;
$R = v/\sigma$	is the generalized measure;
t	is the temperature;
τ	is the time;
x	is the spatial coordinate;
n	is the normal to the surface;
c	is the specific volume heat capacity;
λ	is the coefficient of thermal conductivity;
α	is the heat transfer coefficient;
$Fo = \lambda\tau/cR^2$	is the Fourier number;
$Bi = \alpha R/\lambda$	is the Biot number;
w	is the power per unit volume;
p	is the power per unit area;
ε	is the eccentricity;
$E(\varepsilon)$	is the complete elliptic integral of the second kind;
∇^2	is the Laplace operator;
Δ	is the determinant;
$(\varphi_i, \varphi_j)^v = 1/v\int_v \varphi_i\varphi_j dv$	is the scalar product with respect to volume;
$(\varphi_i, \varphi_j)^\sigma = 1/\sigma\int_\sigma \varphi_i\varphi_j d\sigma$	is the scalar product with respect to area;
δ_{ij}	is the Kronecker symbol;
$\dot{=}$	denotes Laplace transform.

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